

Symmetries and singularities in Hamiltonian systems

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Abstract. This paper contains several results concerning the role of symmetries and singularities in the mathematical formulation of many physical systems. We concentrate in systems which find their mathematical model on a symplectic or Poisson manifold and we present old and new results from a global perspective.

1. Introduction

Many physical problems can be formulated as mathematical problems in the language of symplectic 2-forms or, more generally, in the language of Poisson bivector fields. It was probably Jacobi who first formulated the equations of motion of a particle in terms of position and momenta of a cotangent bundle. He also observed that the set of functions in this manifold is endowed naturally with a bracket, called later the Poisson bracket after Denis Simon Poisson.

A cotangent bundle is naturally endowed with a symplectic structure for which Hamilton's equations correspond simply to the differential equation of the vector field associated to a privileged function called Hamiltonian function (which is a conserved quantity).

The Hamiltonian function of a Hamiltonian system is an integral of motion and the flow of the Hamiltonian vector field (and therefore, the trajectory of the particle) lies on the set $\{H = ct.\}$. This fact is known as Noether's theorem or conservation of energy. Noether's principle for detecting symmetries is based in the preservation of extra functions associated to our system. The set of first integrals have to satisfy special rules; those of complete integrability in the commutative or non-commutative case.

A first example of optimal application of Noether's principle is that of completely integrable systems (in the commutative case). Assume that we have extra $n - 1$ functions (where $2n$ is the dimension of the manifold) which pairwise Poisson commute and are independent; then there is a generalization of Noether's principle to the set of n functions $F = (f_1, \dots, f_n)$ with $f_1 = H$ which says that indeed the particle moves on the invariant set $\{p \in M^{2n}, F(p) = (c_1, \dots, c_n)\}$. Under regularity assumptions, this set is an open submanifold of a cylinder.

In the case the flows of the Hamiltonian vector fields X_{f_i} are complete, the commutation of the flow gives a natural action of \mathbb{R}^n . By identification of the orbit with the quotient of \mathbb{R}^n with its isotropy group we can prove that this manifold is an n -dimensional manifold which indeed is a generalized cylinder $\mathbb{R}^{n-k} \times \mathbb{T}^k$. When the orbit is compact, we obtain the classical Liouville tori which foliate a neighbourhood of the orbit. As a consequence completely integrable systems carry a lot of symmetry (there is an induced action of the cylinder $\mathbb{R}^{n-k} \times \mathbb{T}^k$ acting as a translation on each tori). The orbits of the system lie on a foliation by tori of our symplectic manifold.

But not only that, there is also a Darboux theorem for those systems which tells us that also the symplectic structure can be expressed in a Darboux form with respect to special coordinates associated to our system (action-angle coordinate). This theorem is classically known as Arnold-Liouville theorem but indeed the formulas for the action functions based on integration along the cycles of the Liouville tori can already be found in the works of Henri Mineur [34], [35] and [36].¹

Observe that in the case the manifold is compact the mapping $F = (f_1, \dots, f_n)$ will have singularities because of the maximum principle on this manifold. This giving non-independence of all the functions on M^{2n} . Thus, we generally impose “generic” independence of the functions which means that the rank of the differential of F is maximal on an open dense set.

Compactness can be seen as a weird topological condition for a physicist but indeed many mechanical systems are shown naturally on a compact manifold after symplectic reduction of a cotangent bundle. This is the case of the Euler equations on the sphere (see for instance [10]).

But what happens with singularities? Physically, singularities of the system correspond to relative equilibria of the system. From a mathematical point of view this answer is not satisfactory enough. These questions immediately arise:

- (i) What is the Geometry hidden in these singularities?
- (ii) How are the invariant sets (orbits)?
- (iii) Do we still get normal form results like the one of Arnold-Liouville-Mineur for those?
- (iv) Can we get applications to stability like in the regular case?

These are the main questions that this review paper addresses. In the first section of this paper we give an overlook over the regular results for integrable systems (in the commutative case) and recall some equivariant versions for general symplectic actions (obtained in [41]). In the second section of this paper we will review some results obtained for non-degenerate singularities. Grosso modo, there is a symplectic Morse-bott result that also concerns the neighbourhood of a whole orbit. There is also an equivariant version of this result for compact group actions preserving the system. Most of these results have been obtained by the author in her thesis [37] or together with Nguyen Tien Zung in [38]. In this section, we also review some examples with Morse singularities showing up in physical examples. We finally give applications of these normal form results that have been recently obtained by the author in [39], [40] and [24].

Another way singularities show up in mechanical systems is in the geometrical form associated to the problem. For instance, if the system is considered on a “union” of symplectic manifolds that foliate a Poisson manifold, then singularities may also arise as singularities of the symplectic foliation (non-regular leaves). This is the case of Gelfand-Ceitlin systems on $\mathfrak{u}(n)^*$. The integrable system lies on each coadjoint orbit of $\mathfrak{u}(n)^*$ but the collection of integrable systems can be, indeed, considered as an integrable system on a Poisson manifold. Another context in which the Poisson formulation seems effective is in the magnetic flow in an homogeneous space). We will also describe these examples here. Equivariant aspects in Poisson geometry are interesting field of study since singularities may show up either as singular symplectic leaves of the symplectic foliation or as fixed points of the Poisson/Hamiltonian actions (in the case of integrable systems as singular points of the moment mapping F).

The existence of action-angle coordinates for integrable systems (commutative and non-commutative systems) has been a recent field of study by the author together with Camille Laurent-Gengoux and Pol Vanhaecke in [27]. We will not spell out these results here instead we invite the reader to consult these references.

¹ Henri Mineur is best known for his works in astronomy (a crater on the moon has his name) and for his participation in the French resistance. However, the action formulas that we can find in his paper related to astronomy have constituted a major contribution in classical mechanics.

For compact group actions (not necessarily abelian), the closer result to “Darboux theorem” that we can get in the Poisson context is an equivariant version of Weinstein’s splitting theorem that asserts that locally a Poisson manifold splits as the Poisson product of a symplectic manifold with dimension equal to the rank of the Poisson bivector field at that point, together with a Poisson transverse structure that vanishes at that point. The proof of this result is tricky and needs extra assumptions mainly because the path method which is so popular in symplectic geometry ([43]) does not seem to work so well in the Poisson context.

2. The symplectic regular case

Assume that we are given a Hamiltonian system given by a Hamiltonian function H on a symplectic manifold (M^{2n}, ω) . Hamilton’s equations correspond to the equations of the flow of the Hamiltonian vector field (denoted in the sequel by X_H) associated to H in such a way that the following equation is satisfied,

$$i_{X_H}\omega = -dH.$$

The main philosophy beyond Noether’s principle [44] is that conserved quantities give rise to symmetries in physical systems and can be encoded as group actions on these manifolds.

For instance, the existence of symmetries allows reduction of degrees of freedom of differential equations given by Hamilton’s equations above.

This reduction of degrees of freedom can be formulated in geometrical terms using “Symplectic Reduction” (Marsden-Weinstein Theorem) for Hamiltonian group actions (see for instance [20].)

We are then given a *Moment Map* $F : M^{2n} \longrightarrow \mathfrak{g}^*$ and the reduced space at $\mu \in \mathfrak{g}^*$ is

$$F^{-1}(\mu)/G_\mu^2.$$

From an algebraic-geometric point of view those quotients are studied in Geometric Invariant Theory [25]. The study of singularities of these quotients is specially relevant from an algebro-geometric point of view. Some of these quotients can be interpreted as moduli spaces. For instance in [5] a symplectic structure is given in the moduli space of flat connections on a Riemann surface using reduction. A more recent generalization of this result to the case of punctured Riemann surfaces is contained in [2].

Recall that given a symplectic structure we can associate to it a Poisson bracket in a natural way via the formula:

$$\{f, g\} = \omega(X_f, X_g).$$

In the case there is a set of n (where n is half of the dimension of the symplectic manifolds) functions commuting with respect to this Poisson bracket and the orbits are compact, the group associated to these set of functions is a torus \mathbb{T}^n which acts by translation on the sets $F = cte$ and therefore the reduced space is just a point. This is a consequence of Arnold-Liouville-Mineur theorem that we explain in the first subsection of this section.

In the second subsection we plunge into the non-abelian group case and recall some equivariant Darboux results and give some rigidity results for Hamiltonian and general symplectic group actions.

² There is a Poisson counterpart of this reduction procedure.

2.1. Abelian symmetries

In June 29th of 1853 Joseph Liouville presented a communication entitled “Sur l’intégration des équations différentielles de la Dynamique” at the “Bureau des longitudes”. In the resulting note [26] he relates the notion of integrability of the system to the existence of n integrals in involution with respect to the Poisson bracket attached to the symplectic form. These systems come to the scene with the classical denomination of “completely integrable systems”. In another language, a particular choice of n -first integrals in involution determines the n components of a moment map $\mathbf{F} : M^{2n} \longrightarrow \mathbb{R}^n$.

A lot of work has been done in the subject after Liouville.

Consider a completely integrable Hamiltonian system. The symplectic gradients of the components of the moment map define an involutive distribution. Assume that the moment map is proper (or instead that the invariant sets are compact). Let L be a regular orbit of this distribution then this orbit is a Lagrangian submanifold. Moreover, it is a torus and the neighbouring orbits are also tori. Those tori are called Liouville tori. This is the topological contribution of a theorem which has been known in the literature as Arnold-Liouville theorem. The geometrical contribution of the above-mentioned theorem ensures the existence of symplectic normal forms in the neighbourhood of a compact regular orbit. The formulae for the action formulas can already be found in the work of Mineur. That is why we will refer to the classical Arnold-Liouville theorem as Liouville-Mineur-Arnold theorem. Let us state the theorem below,

Theorem 2.1 (Liouville-Mineur-Arnold Theorem)

Let (M^{2n}, ω) be a symplectic manifold and let $F : M^{2n} \longrightarrow \mathbb{R}^n$ be a proper moment map. Assume that the components f_i of F are pairwise in involution with respect to the Poisson bracket associated to ω and that $df_1 \wedge \cdots \wedge df_n \neq 0$ on a dense set. Let $N = F^{-1}(c)$, $c \in \mathbb{R}^n$ be a connected levelset. Then there exists a neighbourhood $U(N)$ of N and a diffeomorphism $\phi : U(N) \longrightarrow D^n \times \mathbb{T}^n$ such that,

- (i) $\phi(N) = \{0\} \times \mathbb{T}^n$.
- (ii) A set of coordinates μ_i in D^n and a set of coordinates β_i in \mathbb{T}^n for which, $\phi^*(\sum_{i=1}^n d\mu_i \wedge d\beta_i) = \omega$.
- (iii) F depends only on $\phi^*(\mu_i) = p_i$ and it does not depend on $\phi^*(\beta_i) = \theta_i$.

The new coordinates p_i obtained are called action coordinates. The coordinates θ_i are called angle coordinates.

Remark 2.2 The orbits of an integrable system in a neighbourhood of a compact orbit are tori. In action-angle coordinates (p_i, θ_i) the foliation is given by the fibration $\{p_i = c_i\}$ and the symplectic structure is Darboux $\omega = \sum_{i=1}^n dp_i \wedge d\theta_i$.

Remark 2.3 Another consequence of this theorem is that there is a natural group action associated to a completely integrable systems in a neighbourhood of a compact regular orbit. It is the action of a torus \mathbb{T}^n which acts by translations on the orbits (invariant manifolds). This action is naturally induced from the action of \mathbb{R}^n given by the complete joint flow. The isotropy group of this action is a discrete lattice \mathbb{Z}^n . Therefore the action of \mathbb{R}^n gives a toric action when restricted to each orbit. But not only this, the periods of this action can be uniformized in such a way that the action really extends to an action by translations on neighbouring tori.

The existence of action-angle coordinates in a neighbourhood of a compact orbit provides a symplectic model for the Lagrangian foliation \mathcal{F} determined by the symplectic gradients of the n component functions f_i of the moment map F . In fact, Liouville-Mineur-Arnold theorem entails a “uniqueness” result for the symplectic structures making \mathcal{F} into a Lagrangian foliation.

In other words, if ω_1 and ω_2 are two symplectic structures defined in a neighbourhood of N for which \mathcal{F} is Lagrangian then there exists a symplectomorphism preserving the foliation, fixing N and carrying ω_1 to ω_2 . This is due to the following observation: Let X_{f_i} be the symplectic gradients of the functions f_i for any $1 \leq i \leq n$, then the Lagrangian condition implies that in fact $\mathcal{F} = \langle X_{f_1}, \dots, X_{f_n} \rangle$, further $\{f_j, f_k\}_i = 0$ where $\{.,.\}_i$ stands for the Poisson bracket attached to ω_i , $i = 1, 2$. Then by virtue of Liouville-Mineur-Arnold theorem there exists a foliation-preserving symplectomorphism ϕ_i taking ω_i to $\omega_0 = \sum_{i=1}^n dp_i \wedge d\theta_i$. In all, the diffeomorphism $\phi_2^{-1} \circ \phi_1$ does the job. It takes ω_1 to ω_2 , it fixes N and it is foliation preserving.

So if the orbit is regular the existence of action-angle coordinates allows to classify the symplectic germs, up to foliation-preserving symplectomorphism, for which \mathcal{F} is Lagrangian in a neighbourhood of a compact orbit. There is just one class of symplectic germs for which the foliation is Lagrangian.

One could look at the problem of existence of action-angle coordinates on the whole manifold. There are topological obstructions to the existence of global action-angle coordinates as it was shown by Duistermaat in [15].

2.2. General symmetries

Symmetries in physical systems can often be encoded in a Lie group action.

Given an action of a compact Lie group G on a smooth manifold M , Bochner's theorem says that we can linearize it in a neighbourhood of a fixed point for the action. Namely, in a neighbourhood of a fixed point the action of a compact Lie group is locally equivalent to the linear action on the tangent space at this point. Equivalence here means that there exists a diffeomorphism intertwining both actions. Bochner's theorem entails local rigidity for compact group actions.

What is the global counterpart to linearization? Global rigidity.

Rigidity is a concept in geometrical analysis that establishes that in some cases "close" objects are equivalent. When we use the expression close it means that we are using a topology in the set of objects that we are comparing. John Mather [30, 31, 32] was the first to study of rigidity of spaces of functions. Any action α , can be considered as a mapping from $G \times M$ on M with additional properties. This is why the topology considered here is the compact open topology in the set of C^k mappings from $G \times M$ on M , $Map(G \times M, M)$.

In the case of global group actions on compact manifolds, Palais [45] proved that two C^1 -close group actions are equivalent. This result can be understood as a result of equivalence of close compact Lie group actions.

The result of Bochner and Palais, can be easily exported to the symplectic case.

Let us recall these results:

Theorem 2.4 (Equivariant Darboux Theorem) *Let $\rho : G \times M \longrightarrow M$ stand for a symplectic group action on (M, ω) and let p be a fixed point for the action. Then there exists coordinates $(x_1, y_1, \dots, x_n, y_n)$ in a neighbourhood of p such that the group action is linear and the symplectic form can be written as $\omega_0 = \sum_i dx_i \wedge dy_i$.*

The method to produce equivariant normal forms from normal forms was first mentioned by Weinstein in [54]. For a complete proof of Darboux equivariant we refer to the work of Chaperon [8]. The main idea of the proof is to use Moser's path method and the average with respect to a Haar measure of the compact group G .

The result of rigidity for symplectic actions was not found in the literature by the author.

The author proved it in [41]. For a more detailed proof look at [42].

Theorem 2.5 *Let ρ_0 and ρ_1 be two C^2 -close symplectic actions of a compact Lie group G on a compact symplectic manifold (M, ω) then there exists a symplectomorphism ϕ satisfying $\rho(g) \circ \phi = \phi \circ \rho_1(g), \forall g \in G$.*

3. The symplectic Morse singular case

What can be said about action-angle coordinates if the completely integrable systems has singularities?

This question is quite natural because singularities are present in many well-known examples of integrable systems. In fact, if the completely integrable system is defined on a compact manifold then the singularities cannot be avoided.

In the singular case, the problem can be posed at three different levels:

- (i) At the orbit level: In the neighbourhood of a compact singular orbit.
- (ii) At a semi-local level: In the neighbourhood of a compact singular leaf.
- (iii) At a global level.

The problem of topological classification of integrable Hamiltonian systems began with Fomenko [18] in some particular cases. Nguyen Tien Zung [57] studied the general case for the semi-local problem for non-degenerate singularities. It turns out that from a topological point of view we have a product-like description of the singularities in terms of the Williamson type.

The condition of non-degeneracy is always present in the works cited above.

But what is “non-degenerate” in this context?

The singularity of the orbit can be described in terms of the singularity of the functions f_i .

Case I. Let us start with the case L is reduced to a point.

Observe that the Poisson bracket induces a Lie algebra structure in the set of functions. Since the functions f_i are in involution with respect to the Poisson bracket, the quadratic parts of the functions f_i commute defining in this way an abelian subalgebra of $Q(2n, \mathbb{R})$ (the set of quadratic forms on $2n$ -variables). In the case the singularity of the functions f_i is of Morse type this subalgebra is indeed a Cartan subalgebra. We call these singularities of non-degenerate type.

The problem of classification of singularities for the quadratic parts of the functions f_i can be therefore converted into the problem of classification of Cartan subalgebras of $Q(2n, \mathbb{R})$. The singularities for the quadratic parts are well-understood thanks to a result of Williamson [56] where Cartan subalgebras of $Q(2n, \mathbb{R})$ are classified. Let us recall its precise statement,

Theorem 3.1 (Williamson)

For any Cartan subalgebra \mathcal{C} of $Q(2n, \mathbb{R})$ there is a symplectic system of coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ in \mathbb{R}^{2n} and a basis f_1, \dots, f_n of \mathcal{C} such that each f_i is one of the following:

$$\begin{aligned}
 & f_i = x_i^2 + y_i^2 && \text{for } 1 \leq i \leq k_e, && \text{(elliptic)} \\
 & f_i = x_i y_i && \text{for } k_e + 1 \leq i \leq k_e + k_h, && \text{(hyperbolic)} \\
 & \begin{cases} f_i = x_i y_{i+1} - x_{i+1} y_i, \\ f_{i+1} = x_i y_i + x_{i+1} y_{i+1} \end{cases} && \text{(focus-focus pair)} && \\
 & && \text{for } i = k_e + k_h + 2j - 1, 1 \leq j \leq k_f &&
 \end{aligned} \tag{3.1}$$

The linear system given by the quadratic parts of the f_i is called the linear model for a singularity.

We may attach a triple of natural numbers (k_e, k_h, k_f) to a non-degenerate singularity p of F , where k_e stand for the number of elliptic components in the linear model, k_h and k_f the number of hyperbolic and focus-focus components in the linear model respectively.

By virtue of Williamson theorem this triple is an invariant of the linear system. That is why this triple is often called the Williamson type of the singularity.

The result of Eliasson in [16] says that an integrable system that has a non-degenerate singularity can be linearized symplectically in a neighbourhood of this singularity.

This means that we can find coordinates in which we can simultaneously write

- The integrable system in a Williamson basis.
- The symplectic form in Darboux form.

Namely,

Theorem 3.2 (Eliasson) *The integrable system at a point of Williamson type (k_e, k_h, k_f) is locally equivalent to its linear model.*

This theorem was stated for all Williamson's type in Eliasson's thesis [16]. Though a complete proof was only found in the completely elliptic case (Williamson type $(k_e, 0, 0)$). In [37], we give a complete proof for the other cases. The proof included in my thesis is based on an argument that uses symplectically orthogonal decomposition via Moser's path method [43].

Remark 3.3 *This result can be seen as a symplectic Morse result. Because we obtain a Morse theorem for all the components of the moment map in coordinates in which the symplectic structure has a Darboux form.*

Observe that Eliasson's theorem can be seen as a symplectic linearization result which ensures that the initial completely integrable system can be taken to the linear system and that the symplectic form can be taken to the standard one. As a byproduct we obtain a multiple differentiable linearization result for n commuting vector fields with singularities of non-degenerate type.

Case II. L is a compact orbit.

What happens now if we have a compact orbit?

We can define a notion of non-degenerate orbits using reduction.

For these orbits we have the following result which determines existence of normal forms for the symplectic form and the integrable system in the neighbourhood of a nondegenerate orbit but first we need to define a *linear model for higher-dimensional orbits in a covering* as follows,

Denote by (p_1, \dots, p_m) a linear coordinate system of a small ball D^m , $(q_1(\text{mod } 1), \dots, q_m(\text{mod } 1))$ a standard periodic coordinate system of the torus \mathbb{T}^m , and $(x_1, y_1, \dots, x_{n-m}, y_{n-m})$ a linear coordinate system of a small ball $D^{2(n-m)}$. Consider the manifold

$$V = D^m \times \mathbb{T}^m \times D^{2(n-m)} \quad (3.2)$$

with the standard symplectic form $\sum dp_i \wedge dq_i + \sum dx_j \wedge dy_j$, and the following moment map:

$$(\mathbf{p}, \mathbf{h}) = (p_1, \dots, p_m, h_1, \dots, h_{n-m}) : V \rightarrow \mathbb{R}^n \quad (3.3)$$

where

$$\begin{aligned} h_i &= x_i^2 + y_i^2 \quad \text{for } 1 \leq i \leq k_e, \\ h_i &= x_i y_i \quad \text{for } k_e + 1 \leq i \leq k_e + k_h, \\ h_i &= x_i y_{i+1} - x_{i+1} y_i \quad \text{and} \\ h_{i+1} &= x_i y_i + x_{i+1} y_{i+1} \quad \text{for } i = k_e + k_h + 2j - 1, \quad 1 \leq j \leq k_f \end{aligned} \quad (3.4)$$

We will refer to this model as the linear model in a covering of the initial neighbourhood. Indeed, we can define the linearized model on the initial neighbourhood of a point when we do not have hyperbolic components.

For each hyperbolic component, we may have an “hyperbolic twist” giving a generator of $\mathbb{Z}/2\mathbb{Z}$ (for details look at the author’s paper with N.T Zung [40]).

The linear model is V/Γ , with an integrable system on it given by the same moment map:

$$(\mathbf{p}, \mathbf{h}) = (p_1, \dots, p_m, h_1, \dots, h_{n-m}) : V/\Gamma \rightarrow \mathbb{R}^n \quad (3.5)$$

(Γ is a finite group that acts freely and symplectically on the linear model in a covering and comes from the hyperbolic twists associated to hyperbolic components).

The proof of the following theorem is contained in [40],

Theorem 3.4 (Eliasson-Miranda-Zung) *The integrable Hamiltonian system is equivalent to the linear model for Williamson type (k_e, k_h, k_f) in a neighbourhood of non-degenerate compact orbit.*

Remark 3.5 *This theorem is telling us that a neighbourhood of a compact non-degenerate orbit of rank k and Williamson type (k_e, k_h, k_f) of an integrable system is equivalent in a finite covering of it to a symplectic product of different types of cells.*

- (i) *A Liouville-Mineur-Arnold cell which consists of $D^k \times \mathbb{T}^k$ with a symplectic form in action-angle coordinates, $\omega = \sum_i dp_i \wedge d\theta_i$.*
- (ii) *A symplectic product of exactly k_e two dimensional cells with elliptic singularities with induced integrable system with moment map $f = x^2 + y^2$ and symplectic form $\omega = dx \wedge dy$.*
- (iii) *A symplectic product of exactly k_h two dimensional cells with elliptic singularities with induced integrable system with moment map $f = xy$ and symplectic form $\omega = dx \wedge dy$.*
- (iv) *A symplectic product of exactly k_f four dimensional cells with focus-focus singularities with induced integrable system with components of the moment map $h_i = x_i y_{i+1} - x_{i+1} y_i$ and $h_{i+1} = x_i y_i + x_{i+1} y_{i+1}$ and symplectic form $\omega = dx_i \wedge dy_i + dx_{i+1} \wedge dy_{i+1}$.*

In the case there are no hyperbolic components, it is exactly equivalent to the symplectic product.

In [40] we ended up proving the equivariant version of this theorem result in a neighbourhood of a singular compact orbit.

This result is really relevant because it encodes extra symmetries of the integrable system. All the symmetries considered are compact.

A nice consequence is the abelianity of the connected component of the identity of the group of symplectomorphisms preserving the moment map. In particular, in the case the action of the group is effective then this group is Abelian, in all, since it is also compact it is a product of a torus with a finite group. In the end, in the case the group is connected we recover actions by tori in the spirit of the theorem of Delzant.

Namely in [40] we proved,

Theorem 3.6 (Miranda-Zung) *The integrable system is equivariantly equivalent in a neighbourhood of a compact orbit of dimension k and Williamson type (k_e, k_h, k_f) to the integrable system given by the linear model endowed with the linear group action.*

Remark 3.7 *The linear group action means transversally linear and acting by translations on Liouville tori. For details look at the paper [40].*

Remark 3.8 *This result can be seen as a symplectic Morse-Bott result. Because we obtain a Morse-Bott theorem (transversally Morse) for all the components of the moment map in coordinates in which the symplectic structure has a Darboux form.*

Indeed we already needed the equivariant version to obtain symplectic equivalence in the initial neighbourhood applied to the discrete group of order two.

3.1. Examples showing up in physical systems and construction of integrable systems with singularities

The above results will have much more value if these kind of singularities are present in simple physical systems. Here we present a list of examples that contain non-degenerate singularities (most of them just non-degenerate). In the inspiring book by Larry Bates and Richard Cushman [10] we can find many more and a close inspection of the symmetries naturally associated to these problems.

The Rotation of a sphere

Consider a 2-sphere spinning around the z -axis. This defines a Hamiltonian system associated to the height function of the sphere. Indeed the natural \mathbb{R} -action is periodic and gives an S^1 -action.

The fixed points for the S^1 -action correspond to singularities of the Hamiltonian function. The singular values are the boundary of the interval $[-1, 1]$ which is the image under the height function. The singular points are the north and the south pole.

Indeed the north and the south poles are really elliptic singularities. (This construction recalls the one of Morse theory [33]).

The harmonic oscillator

The harmonic oscillator correspond to the movement of a particle following the solution of the differential equations.

$$\ddot{x} = -ax$$

The equation above is known as Hooke's law and corresponds to the physical situation of a spring-mass system.

The symplectic structure is the one of the cotangent bundle of \mathbb{R} .

By conservation of energy the Hamiltonian of the system is the sum of Kinetic and potential energy which in position and momenta coordinates reads as,

$$H = \frac{1}{2}(x^2 + ay^2).$$

The origin is the only singular point and the singularity is of elliptic type. Indeed, for $a = 1$ we get exactly Eliasson's normal form.

The Euler's equations on the sphere

The Euler's equations on the sphere correspond to the movement of the Euler top.

Let us quickly review this example. Take our initial symplectic manifold to be $T^*(SO(3, \mathbb{R}))$. The action of $SO(3, \mathbb{R})$ on itself by left translations can be lifted to the cotangent bundle. Recall that the lift of any action of a manifold to its cotangent bundle is Hamiltonian.

In the sequel, we closely follow [10] and notation therein.

The reduced symplectic spaces $F^{-1}(\mu)/SO(3, \mathbb{R})_\mu$ coincide with the coadjoint orbits on $\mathfrak{so}(3)^*$ endowed with the Kostant-Kirillov-Souriau symplectic form. Identifying $\mathfrak{so}(3)$ with (\mathbb{R}^3, \wedge) , the coadjoint orbits are spheres and the Kostant-Kirillov-Souriau form is the standard area form.

The reduced Euler equations are the Hamiltonian equations on these reduced symplectic spaces coming from the following Hamiltonian on $T^*(SO(3))$:

$$\begin{aligned} H : T^*(SO(3)) &\longrightarrow \mathbb{R} \\ \alpha_A &\longmapsto \frac{1}{2}\rho(A)(\alpha_A, \alpha_A) \end{aligned}$$

(where α_A denotes an element in $T^*(SO(3))$ according to the standard trivialization of the cotangent bundle of a Lie group and $\rho(A)$ is a left invariant metric on $T^*(SO(3))$.)

In [10], Cushman and Bates use this trivialization of $T^*(SO(3))$ to write down the Hamiltonian in terms of the *moments of inertia*, and find the reduced Euler equations in the reduced space. In section III.4 in their book, they show that this system has exactly two hyperbolic singularities and four elliptic singularities.

To see a beautiful picture of this system and their singularities look at the cover of the book by Marsden and Ratiu where we can appreciate very clearly the dynamics of these singularities [29].

The spherical pendulum

Consider the movement of a point that represents a ball attached to a fixed point. This gives the simple pendulum.

If we consider the movement of this ball constrained to a sphere, we obtain the spherical pendulum.

The phase space is $T^*(S^2)$. The manifold being four-dimensional in order to prove integrability we need to show that there exists two first integrals in involution.

One of the first integrals is given by the complete energy of the system and the second one is given by the existence of an additional symmetry given by rotation around the z -axis.

This system can be shown to be completely integrable see for instance [10] and it has two critical points one of hyperbolic type and one of focus-focus type.

Construction of integrable systems

Bolsinov and Fomenko give in their book [19] a procedure to include singularities of different type of non-degenerate via surgery adapted to the foliation given by the integrable systems.

In the case of compact surface their method is very simple to describe and can be done by changing strips with regular foliations by strips containing one hyperbolic and one elliptic singularity. This contribution does not change the Euler characteristic as expected.

Together with Mark Hamilton, we gave given a precise description of this procedure in [24].

4. Applications of Normal forms

The theorems listed above in the regular and singular case are normal form theorems for the symplectic form and the integrable system.

As such it has many applications. We explicit a pair of applications here. Another beautiful application of these normal form results is the computation of topological entropy of Hamiltonian system. This is a joint work with Jean-Pierre Marco [28] but we will not spell it out here.

4.1. Stability of integrable systems

The stability consequences of the Liouville-Mineur-Arnold theorem have been largely explored.

The normal form results for non-degenerate singularities (points or orbits) implies stability in the set of moment maps.

Indeed we also obtain infinitesimal stability. This is a result that we have obtained together with San Vu Ngoc [39].

An infinitesimal deformation of an integrable system $F = (f_1, \dots, f_n)$ is given by a set of n functions g_i that modulo ε^2 satisfy ,

$$\{f_i + \varepsilon g_i, f_j + \varepsilon g_j\} \equiv 0.$$

We can view these infinitesimal deformations as 1-cocycles of a complex (deformation complex) associated to the integrable system [39].

Given integrable system F we can introduce a deformation complex by considering the adjoint representation of \mathbb{R}^n on $C^\infty(M)$ given by the Poisson bracket. This complex is defined via the Chevalley-Eilenberg cohomology.

Together with San Vu Ngoc, we proved in [39] an infinitesimally stability result which corresponds to the vanishing of the first cohomology group of this complex.

Indeed the vanishing of the first cohomology group is given via a singular Poincaré lemma which proves that every 1-cocycle is a coboundary.

Theorem 4.1 (Miranda, San Vu Ngoc) *Let q_1, \dots, q_n be a standard basis (in the sense of Williamson) of a Cartan subalgebra of $\mathcal{Q}(2n, \mathbb{R})$. Then the corresponding completely integrable is smoothly infinitesimally stable at $m = 0$: that is,*

$$H^1(\mathfrak{q}) = 0.$$

4.2. Applications to geometric quantization

Geometric quantization associated to a real polarization on a symplectic manifold (M^{2n}, ω) with a prequantum line bundle can be given via the cohomology groups with coefficients in the sheaf of flat section \mathcal{J} . This geometric quantization is often called Kostant's quantization.

Real polarizations are just given by Lagrangian foliations on the symplectic manifold. Leaves that admit sections that are flat along them are called Bohr-Sommerfeld leaves.

In the case the real polarization is fibrating these computations can be carried using sheaf cohomology. Sniatycki proved in [46] the following result:

Theorem 4.2 *In the case the polarization is fibrating and the fibers are compact all the cohomology groups $H^i(M, \mathcal{J})$ vanish except for the one in dimension n . This cohomology group is determined by the Bohr-Sommerfeld leaves.*

In the paper [21] by Guillemin and Sternberg this result is reinterpreted in terms of action-angle coordinates. More precisely, they construct action coordinates on the basis of the fibration and identify the Bohr-Sommerfeld leaves with the ones whose action functions are integer-valued.

Together with Mark Hamilton, we have considered a real polarization given by an integrable systems on a compact manifolds with non-degenerate singularities. We have used the singular normal forms explained before to compute the Geometric Quantization in the surface case [24].

5. Symmetries in Poisson manifolds

Many Hamiltonian systems have their natural context in the Poisson setting. The study the symmetries for these systems becomes a challenging thing since we have extra contribution of singularities which comes from the singularities of the symplectic foliation associated to the Poisson vector field.

Some of them are integrable (in the commutative or non-commutative sense). We will show some examples in the second subsection of this section.

5.1. General symmetries

In this section we describe the local structure of the Poisson manifolds under the existence of symmetries.

There is no Darboux theorem for Poisson bivector fields. The best that we can achieve is a splitting theorem due to Weinstein.

Theorem 5.1 (Weinstein) *Let (P^n, Π) be a smooth Poisson manifold and let p be a point of P of rank $2k$, then there is a smooth local coordinate system $(x_1, y_1, \dots, x_{2k}, y_{2k}, z_1, \dots, z_{n-2k})$ near p , in which the Poisson structure Π can be written as*

$$\Pi = \sum_{i=1}^k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

where f_{ij} vanish at the origin.

This result is telling us that the Poisson manifold is locally a product of a symplectic manifold with a Poisson manifold with vanishing Poisson structure at the point.

$$(P^n, \Pi, p) \approx (M^{2k}, \omega, p_1) \times (P_0^{n-2k}, \Pi_0, p_2)$$

With Nguyen Tien Zung we have proved the equivariant version of this result under some mild assumptions (“tameness”) on the Poisson structure.

The result is contained in [38].

Theorem 5.2 (Miranda-Zung) *Any tame Poisson manifold admits an equivariant splitting for an action of a compact Lie group.*

Remark 5.3 *This equivariant splitting is obtained using Vorobjev coupling [50] associated to a Poisson fibration. This method extends the minimal coupling method of Sternberg [47] and geometric data associated to Symplectic fibration to the Poisson context.*

In the case the transverse Poisson structure has linear part of semisimple type we can combine the linearization result of Conn [9] together with this equivariant splitting to obtain an equivariant linearization result contained in [38].

We have also obtained some rigidity results similar to those of Palais in the Poisson context. These are contained in the recent work [42].

In the case of abelian symmetries or symmetries given by an integrable system, we have obtained an extension of the action-angle theorem to the Poisson context together with Camille Laurent-Gengoux and Pol Vanhaecke in [27]. The reader is invited to consult this reference for further information.

5.2. Examples of integrable system in the Poisson context

In this section we present two examples that justify the interest of integrable systems in the Poisson context. With Alain Albouy we have been recently looking at another wide source of examples which come from Projective Dynamics and use Appel transformations. Other “more classical” examples include,

The Gelfand-Ceitlin system

The Gelfand-Cetlin system has been classically [20] considered as an integrable system on each coadjoint orbit \mathcal{O} of $\mathfrak{u}(n)^*$.

On the other hand the collection of coadjoint orbits gives the Poisson manifold $\mathfrak{u}(n)^*$ and the Gelfand-Ceitlin system can be seen as an integrable system in the Poisson sense.

This example is interesting because the dual of a Lie algebra's constitute one of the basic examples of Poisson manifolds.

Let us recover this system in another way.

- (i) Consider the increasing sequence of Lie algebra inclusions which comes from considering the Lie algebra $\mathfrak{u}(k)$ as the left-upper diagonal block of $\mathfrak{u}(k+1)$,

$$\mathfrak{u}(1) \subset \cdots \subset \mathfrak{u}(n-1) \subset \mathfrak{u}(n)$$

- (ii) We can dualize the inclusions to get a sequence of surjective mappings (the projections) which are indeed Poisson maps:

$$\mathfrak{u}(n)^* \longmapsto \mathfrak{u}(n-1)^* \longmapsto \cdots \longmapsto \mathfrak{u}(1)^*$$

According to the method of Thimm (see for instance [20]), if we consider the generators of the Casimirs of all $\mathfrak{u}(k)^*$ for $k = 1, \dots, n$ and we pull them back to $\mathfrak{u}(n)^*$, we then obtain a set of functions which Poisson commute and define an integrable system. For particular choice of generators (as the ones described in [21], its restriction to an open subset of \mathcal{O} gives the Gelfand-Cetlin system.

This system is defined not only when restricted to the coadjoint orbit with the Kirillov-Kostant-Souriau symplectic structure but on $\mathfrak{u}(n)^*$.

In [21] Guillemin and Sternberg compute the geometric quantization of the Gelfand-Ceitlin systems via Sniatycki's result that compares geometric quantization with Bohr-Sommerfeld orbits.

Sniatycki's result yields interesting results about representations of $U(n)$ as can be seen in [21]. As we mentioned before, the Bohr-Sommerfeld leaves of an integrable system which is fibrating (when restricted to each coadjoint orbit) can be computed once we obtain a whole

set of action functions (when restricted to each Bohr-Sommerfeld leaf). The Bohr-Sommerfeld leaves are then identified as the leaves for which the moment map (when restricted to each Bohr-Sommerfeld leaf) has integral value. The theorem of Sniatycki for Gelfand-Ceitlin systems just gives information about irreducible representations of $U(n)$ (see Theorem 6.1 in [21]). Let us point out here that in all these computations singularities are avoided.

Magnetic flows on Homogeneous spaces

The geodesic flow can be interpreted as the inertial motion of a particle on a Riemannian manifold (Q, g_{ij}) with kinetic energy given by:

$$H(x, p) = \frac{1}{2} \sum_{i,j} g^{ij} p_i p_j$$

The symplectic structure associated to g comes from the cotangent symplectic structure with momenta defined via $p_j = g_{ij} \dot{x}_i$. The motion of a particle under additional magnetic field given by a closed 2-form on Q :

$$\Omega = \sum_{i,j} F_{ij}(x) dx_i \wedge dx_j$$

is described by the Hamilton equations associated to the symplectic structure $\omega + \rho^*(\Omega)$ where ρ is the standard projection from T^*Q to Q .

In the sequel, we follow the paper [7].

Let G be a compact Lie group and H a closed subgroup. Let $a \in \mathfrak{h}$ be H -adjoint invariant.

In particular, $H \in G_a$. Let $\mathcal{O}(a)$ stand for the adjoint orbit. We then have a submersion of homogeneous spaces $\sigma : G/H \rightarrow G/G_a \cong \mathcal{O}(a)$ and $\omega = \sigma^*(\Omega_{KKS})$ gives a magnetic field on G/H .

We then have,

Theorem 5.4 (Bolsinov, Jovanovich) *The magnetic geodesic flows of normal metric ds_0^2 in G/H with respect to the magnetic form ω is completely integrable in the non-commutative sense.*

By using a Theorem of Mischenko-Fomenko which says that non-commutative integrability implies commutative integrability) we obtain integrability of the magnetic geodesic flows on G/H . Again, this can be viewed in the Poisson manifold $T^*(G)/H$.

6. Acknowledgments

The author is thankful to the organizers of this conference for having gathered people with complementary interests in the field which made the conference really enriching. The author is currently supported by a Juan de la Cierva contract reference number JCI-2005-1712-18 and partially supported by the DGICYT/FEDER, project number MTM2006-04353 (Geometría Hiperbólica y Geometría Simplética).

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